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THE METRICAL INTERPRETATION OF SUPERREFLEXIVITY IN BANACH SPACES

BY

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ABSTRACT

The main result is a metrical characterization of superreflexivity in Banach spaces. A Banach space X is not superreflexive if and only if X contains hyperbolic trees as a metric space. The notion of non-linear cotype in discussed.

1. Introduction

If follows in particular from Ribe's result [8], stating that uniformly homeomorphic Banach spaces are finitely representable in each other, that the notions from local theory of normed spaces are determined by the metric structure of the space and thus have a purely metrical formulation. The next step consists in studying these metrical concepts in general metric spaces in an attempt to develop an analogue of the linear theory. A detailed exposition of this program will appear in J. Lindenstrauss's forthcoming survey paper [5]. We also refer the reader to this paper and to [6] for notions which are not defined here. Recall that in our "dictionary" linear operators are translated in Lipschitz maps, the operator norm by the Lipschitz constant of the map

$$\|f\|_{\mathrm{Lip}} = \sup_{x \neq y} \frac{\operatorname{dist}(F(x), F(y))}{\operatorname{dist}(x, y)}$$

where "dist" is taken in the appropriate metric space. The translations of "Banach–Mazur distance" and "finite-representability" in linear theory are immediate. At the roots of the local theory of normed spaces are properties such as type, cotype, superreflexivity, ... related to the geometry of the unit ball. The analogue of type in the geometry of metric spaces is the fact that Hamming cubes are not uniformly embeddable in the given metric space. This result was proved

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and elaborated in [2]. A simple metrical invariant replacing the notion of cotype was not yet discovered. A natural construction indicated to the author by W. B. Johson consists in looking at the Lipschitz-dual of the metric space X, d which is the Banach space Lip(X) of real-valued Lipschitz functions on X with norm

$$\|f\|_{\mathrm{Lip}(X)} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}$$

and to study the cotype of the dual space $(\mathrm{Lip}\, X)^*$. An immediate difficulty with this approach is the fact that the geometry of spaces of Lipschitz functions is very poorly understood and in particular it is an unsolved problem whether the dual of the space of Lipschitz functions on the square $[0,1] \times [0,1]$ has a finite cotype. However, in an appendix to this paper, we prove that $(\mathrm{Lip}\, X)^*$ has no bounded cotype when X is taken to be the Hamming cube. Since the Hamming cubes are the metrical analogues of the l_n^1-spaces ($n = 1, 2, \dots$), this observation means that the previous approach is not satisfactory in this simple form. The main result of the paper is the metrical substitute for superreflexivity. For $J = 1, 2, \dots$, denote $\Omega_j = \{1, -1\}^j$ and $T_j = \bigcup_{j' \leqq j} \Omega_{j'}$, $T = \bigcup_{j=1}^{\infty} T_j$. Thus T_j is the finite tree with j levels and T the infinite tree. The graph on T corresponding to the tree-structure induces a metric ρ (the hyperbolic distance). The metric on T_j is the restriction of ρ.

THEOREM 1. *A Banach space X is not superreflexive if and only if the trees T_j admit a uniform Lipschitz embedding in X, thus iff t is finitely represented in X as metric space.*

Let M, ρ be a compact metric space. We define the corresponding hyperbolic space $\tilde{M}$, $\tilde{\rho}$ by $\tilde{M} = M \times \{1, 2, 3, \dots\}$ with distance

$$\tilde{\rho}((x, t), (x', t')) = |t = t'| + \sum_{0 \leqq s \leqq \min(t, t')} \min(1, 2^s \rho(x, x')).$$

COROLLARY 2. *If M is infinite, $\tilde{M}$ is embeddable in no superreflexive space.*

This result originates in the problem raised by M. Gromov (personal communication) whether or not there is Lipschitz embedding of the hyperbolic tree in Hilbert space. The author is indebted to him for some valuable discussions about these questions.

In the next sections we prove the necessity and sufficiency of the condition stated in Theorem 1. The proofs are particularly simple and make essential use of characterizations of superreflexivity of Banach spaces due to R. C. James

([3], [4]) and G. Pisier [7]. The last section is the remark related to non-linear cotype which we indicated earlier.

2. Non-embeddability of trees in superreflexive space

The argument is an adaptation of the proof in case of Hilbert space ([1]). Recall the following fact ([7], Lemma 3.1).

LEMMA 1. *Given a superreflexive space X, there is $p<\infty$ and $C<\infty$ such that if $(\xi_s)_{s=1,2,\dots}$ is an X-valued martingale on some probability space Ω, then*

$$\Sigma_s \| \xi_{s+1} - \xi_s \|_p^p \leqq C \sup_s \| \xi_s \|_p^p \tag{1}$$

where $\| \ \|_p$ stands for the norm in $L^p_X(\Omega)$.

Actually, our martingales will be standard diadic Walsh–Paley martingales. The proof of Lemma 1 uses the observation that the sequence $\xi_2-\xi_1, \xi_3-\xi_2, \dots, \xi_{s+1}-\xi_s, \dots$ is a basic sequence in the space $L^p_X(\Omega)$ and R. C. James' estimates on basic sequences in superreflexive spaces [3, 4]. Notice that if X is a Hilbert space, then $p=2$, $C=1$.

Lemma 1 is used to prove

LEMMA 2. *If $x_1,\dots,x_J$ is a finite system of vectors in X, then with previous notations*

$$\inf_{d,\, d\leqq j\leqq J-d} \frac{1}{d} \| 2x_j - x_{j-d} - x_{j+d} \| \leqq C(\log J)^{-1/p} \sup_{1\leqq j\leqq J} \| x_{j+1} - x_j \| . \tag{2}$$

PROOF. Denote $\mathscr{D}_0 \subset \mathscr{D}_1 \subset \cdots \subset \mathscr{D}_r$ the algebras of intervals on $[0,1]$ obtained by successive diadic refinements, letting $J=2^r$. Define the X-valued function

$$\xi = \sum_{1\leqq j\leqq J-1} \chi_{[j/J,(j+1)/J[}(x_{j+1}-x_j), \qquad \chi = \text{indicator function}$$

and consider the expectations $\xi_s = \mathbf{E}[\xi \mid \mathscr{D}_s]$ for $s=1,\dots,r$. Since ξ_s form a martingale ranging in X, Lemma 1 implies

$$C\|\xi\|_p^p \geqq \sum_{s=1}^{r} \| \xi_{s+1} - \xi_s \|_p^p \tag{3}$$

where by construction

$$\| \xi \|_p \leqq \| \xi \|_\infty = \sup_j \| x_{j+1} - x_j \| \tag{4}$$

and clearly

$$\| \xi_{s+1} - \xi_s \|_p^p = 2^{-r+s} 2^{-ps} \sum_{1 \leqq t \leqq 2^{r-s}} \| 2x_{t.2^s} - x_{(t-1)2^s} - x_{(t+1)2^s} \|^p$$
$$\geqq 2^{-ps} \min_{2^s \leqq j \leqq J-2^s} \| 2x_j - x_{j-2^s} - x_{j+2^s} \|^p.$$

Combined with (3), (4), it follows that

$$C \sup_j \| x_{j+1} - x_j \| \geqq r^{1/p} \min_{1 \leqq s \leqq r} \min_{2^s \leqq j \leqq J-2^s} (2^{-s} \| 2x_j - x_{j-2^s} - x_{j+2^s} \|)$$

and hence inequality (2).

LEMMA 3. *Denote* Ω *a probability space and* $f_1, \ldots, f_J$ *a sequence of functions in* $L_X^\infty(\Omega)$. *Then*

$$\inf_{d, d \leqq j \leqq J-d} \frac{1}{d} \| 2f_j - f_{j-d} - f_{j+d} \|_p \leqq C(\log J)^{-1/p} \sup_{1 \leqq j \leqq J} \| f_{j+1} - f_j \|_\infty. \tag{5}$$

PROOF. Replace X by $L_X^p(\Omega)$, for which (1) remains valid, and use (2).

LEMMA 4. *Let* $f_1, \ldots, f_J$ *be* X*-valued functions on* $\{1, -1\}^N$ *where* f_j *only depends on* $\varepsilon_1, \ldots, \varepsilon_j$. *Then there exists* d, $1 \leqq j \leqq J - d$ *and* $\varepsilon \in \Omega_j \equiv \{1, -1\}^j$ *such that*

$$\iint_{\Omega_d \times \Omega_d} \| f_{j+d}(\varepsilon, \delta) - f_{j+d}(\varepsilon, \delta') \| \, d\delta d\delta' \leqq Cd(\log J)^{-1/p} \sup_j \| f_{j+1} - f_j \|_\infty. \tag{6}$$

PROOF. By (5), there is d and $d \leqq j \leqq J - d$ such that

$$\| 2f_j - f_{j+d} - f_{j-d} \|_p \leqq Cd(\log J)^{-1/p} \sup_j \| f_{j+1} - f_j \|_\infty.$$

Write explicity

$$\| 2f_j - f_{j-d} - f_{j+d} \|_p^p = \iint_{\Omega_j \times \Omega_d} \| 2f_j(\varepsilon) - f_{j-d}(\varepsilon) - f_{j+d}(\varepsilon, \delta) \|^p \, d\varepsilon d\delta$$

implying

$$2^p \| 2f_j - f_{j-d} - f_{j+d} \|_p^p \geqq \iiint_{\Omega_j \times \Omega_d \times \Omega_d} \| f_{j+d}(\varepsilon, \delta) - f_{j+d}(\varepsilon, \delta') \|^p \, d\delta d\delta' d\varepsilon,$$

$$\| 2f_j - f_{j-d} - f_{j+d} \|_p \geqq \frac{1}{2} \min_{\varepsilon \in \Omega_j} \iint_{\Omega_d \times \Omega_d} \| f_{j+d}(\varepsilon, \delta) - f_{j+d}(\varepsilon, \delta') \| \, d\delta d\delta',$$

thus (6).

We are now ready to prove the first part of Theorem 1. Fix J and consider a one-to-one map $F\colon T_J \to X$. Apply Lemma 4 to the functions $f_1, \dots, f_J$ defined by

$$f_j(\varepsilon) = F(\varepsilon \mid j) \qquad \text{for } \varepsilon \in \{1, -1\}^J.$$

Notice that by definition of the metric ρ on T_J

$$\sup_j \| f_{j+1} - f_j \|_\infty = \sup_{\varepsilon \in \Omega_J} \sup_{1 \leqq j < J} \| F(\varepsilon \mid j+1) - F(\varepsilon \mid j) \| \leqq \| F \|_{\mathrm{Lip}}$$

and similarly

$$\| f_{j+d}(\varepsilon, \delta) - f_{j+d}(\varepsilon, \delta') \| \geqq \| F^{-1} \|_{\mathrm{Lip}}^{-1} \rho((\varepsilon, \delta), (\varepsilon, \delta')).$$

Therefore

$$Cd(\log J)^{-1/p} \| F \|_{\mathrm{Lip}} \| F^{-1} \|_{\mathrm{Lip}} \geqq \iint_{\Omega_d \times \Omega_d} \rho((\varepsilon, \delta), (\varepsilon, \delta')) d\delta d\delta' \sim d$$

applying (6). Thus we get the following estimate on the distorsion of F,

(7) $$\operatorname{dist}(F) \gtrsim (\log J)^{1/p}.$$

Thus in case of Hilbert space

(8) $$\operatorname{dist}(F) \gtrsim (\log J)^{1/2}.$$

As we will show in the next section, (8) is an optimal result.

3. Embedding hyperbolic trees in non-superreflexive spaces

If conversely X is not superreflexive, it is known that for arbitrarily large $J = 1, 2, \dots$, there are vectors $x_1, x_2, \dots, x_K$ in the unit ball of X satisfying the condition

(9) $$\inf_{1 \leqq j \leqq K} \operatorname{dist}[\operatorname{conv}(0, x_1, \dots, x_j), \operatorname{conv}(x_{j+1}, \dots, x_K)] > \tfrac{1}{2}.$$

Let $\Phi\colon T_J \to \{1, 2, \dots, 2^{J+1} - 1\}$ be an enumeration such that any pair of segments in T_J starting at incomparable nodes (with respect to the tree ordering) are mapped inside disjoint intervals.

Take $x_1, \dots, x_{2^{J+1}}$ satisfying (9) and define $F\colon T_J \to X$ by the formula

$$F(\varepsilon) = \sum_{j' \leqq j} x_{\Phi(\varepsilon \mid j')}, \qquad \varepsilon \in \Omega_j, \quad j \leqq J.$$

The reader will easily check that F has bounded distorsion. This completes the proof of Theorem 1.

In the case $X = H =$ Hilbert space, an embedding of T_J in H with distorsion $\sim (\log J)^{1/2}$ can be constructed in the following way. Let $\{e_\varepsilon \mid \varepsilon \in T_J\}$ be an orthonormal system in H and let for $j \leqq J$, $\varepsilon \in \Omega_j$

$$F(\varepsilon) = \sum_{j' \leqq j} (j - j' + 1)^{1/2} e_{\varepsilon|j'}. \tag{10}$$

It is then easily seen that the restriction of F to each level Ω_j in T_j is of bounded distorsion. However, the distorsion on branches of T_J is of order $(\log J)^{1/2}$, since for $\varepsilon \in \Omega_J$

$$\|F(\varepsilon) - F(\varepsilon \mid J-1)\|_H^2 = 1 + \sum_{j<J} [(J-j+1)^{1/2} - (J-j)^{1/2}]^2 \sim \log J.$$

REMARK. The level sets of T, ρ embed in $\mathbf{R}, d$ where

$$d(x,y) = \log(1 + |x-y|).$$

Now the metric space $\mathbf{R}, d$ embeds q-conformally in Hilbert space since d is quasi-concave with respect to the usual distance $|x-y|$ (see P. Assouad's thesis, Chapter IV for definitions and details).

4. Appendix: Remark on the Lipschitz dual of the Hamming cube

If X, d is a metric space, denote $\mathrm{Lip}(X)$ the Banach space of Lipschitz functions f on X with norm

$$\|f\|_{\mathrm{Lip}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}.$$

Notice that by the extension theorem for Lipschitz functions, given a subset Y of X endowed with the induced metric, the restriction map gives a quotient map $q_Y: \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$. The n-Hamming cube is the set $\Omega_n = \{1, -1\}^n$ endowed with the metric $\rho(\varepsilon, \varepsilon') = \sum_{j=1}^n |\varepsilon_j - \varepsilon'_j|$. We will prove the following fact:

PROPOSITION 3. $\mathrm{Lip}(\Omega_n)$ *contains uniformly complemented* l_k^1*-subspaces when* $n \to \infty$. *Hence* $\mathrm{Lip}(\Omega_n)^*$ *has no uniformly bounded cotype for* $n \to \infty$.

Let us agree to denote for a given metric space Z, ρ by Z^n the n-fold product space with metric $\rho(\bar{x}, \bar{y}) = \sum_{1 \leq j \leq n} \rho(x_j, y_j)$. If Z is a two-point set, we find the Hamming cube back.

Denote $\mathbf{Z}_k$ the cyclic group $\mathbf{Z}/k\mathbf{Z}$ of residue classes mod k endowed with the natural metric (which is the word-metric with respect to the generator $e^{2\pi i/k}$). We show that for $k = 1, 2, \ldots$ there is a Lipschitz embedding of $\mathbf{Z}_{2k}$ into Ω_k with distorsion 1. Hence, by a previous remark, the spaces Ω_n in Proposition 3 may be replaced by the Hamming products $[\mathbf{Z}_k]^n$ or, involving a discretization argument, by the n-fold torus Π^n equipped with metric

$$\rho(\bar{\theta}, \bar{\psi}) = \sum_{1 \leq j \leq n} | e^{i\theta_j} - e^{i\psi_j} |.$$

At this point, we are thus led to consider the Lipschitz space $\mathrm{Lip}(\Pi^n)$. Notice that convolution with kernels of the form

$$F(\bar{\theta}) = \prod_{j=1}^{n} F_N(\theta_j), \qquad F_N(\theta) = \sum_{|j| \leq N} \frac{N - |j|}{N} e^{ij\theta} = \text{Féjer Kernel}$$

gives operators c_N: $\mathrm{Lip}(\Pi^n) \to W^{1,\infty}_{\infty}(\Pi^n)$ satisfying

$$\lim_{N \to \infty} c_N i = \mathrm{Id} \text{ pointwise}$$

where $W^{1,\infty}(\Pi^n) \equiv W$ is the Sobolev space of functions f on Π^n with bounded partial derivatives $\partial_j f = \partial f / \partial \theta_j$ and norm

$$\| f \|_W = \sup_{1 \leq j \leq n} \| \partial_j f \|_\infty ,$$

and $i = W \to \mathrm{Lip}(\Pi^n)$ is the identity operator (isometric embedding). It results from this discussion that Proposition 3 is a consequence of the following two facts:

LEMMA 5. *$\mathbf{Z}_{2k}$ embeds in Ω_k.*

LEMMA 6. *$W^{1,\infty}_{\infty}(\Pi^n)$ contains uniformly complemented l^1_k-subspaces for $n \to \infty$.*

PROOF OF LEMMA 5. Consider the map $F : \{1, e^{2\pi i/2k}, \ldots, e^{2\pi i(2k-1)/2k}\} \to \Omega_k$ given by

$$F(e^{2\pi i\alpha/2k}) = (\underbrace{1, \ldots, 1}_{\alpha}, -1, \ldots, -1) \qquad \text{if } \alpha \leq k$$

and

$$F(e^{2\pi i\alpha/2k}) = (\underbrace{-1, \ldots, -1}_{\alpha - k}, 1, \ldots, 1) \qquad \text{if } \alpha > k.$$

Clearly

$$\rho(F(e^{2\pi i\alpha/2k}), F(e^{2\pi i\alpha'/2k})) \sim k \, | e^{2\pi i\alpha/2k} - e^{2\pi i\alpha'/2k} |.$$

PROOF OF LEMMA 6. This is the heart of the matter. Fix an integer J and an enumeration $\sigma^{(s)}$ $(1 \leqq s \leqq 2^J)$ of Ω_J. Consider a rapidly increasing sequence k_j and put

$$A(\bar{\theta}) = \prod_{1 \leqq j \leqq J} \left[1 + \cos k_j \left(\sum_{s=1}^{2^J} \sigma_j^{(s)} \theta_s \right) \right].$$

Thus $A \geqq 0$ and $\int A(\bar{\theta}) d\theta = 1$ by the choice of the sequence k_j. Take f in W. Then

$$\| f * A \|_W \leqq \| f \|_W \tag{1}$$

and $f * A$ has the form

$$f * A = \hat{f}(0) + \sum_{j=1}^{J} \left\{ D_{j,+}(\bar{\theta}) \exp \left[2\pi i k_j \left(\sum_{s=1}^{2^J} \sigma_j^{(s)} \theta_s \right) \right] + D_{j,-}(\bar{\theta}) \exp \left[-2\pi i k_j \left(\sum_{s=1}^{2^J} \sigma_j^{(s)} \theta_s \right) \right] \right\}$$

where $D_{j,+}$, $D_{j,-}$ have Fourier transform supported by

$$\left\{ \left(\sum_{i<j} \varepsilon_i \sigma_i^{(1)} k_i, \ldots, \sum_{i<j} \varepsilon_i \sigma_i^{(2^J)} k_i \right) \,\middle|\, \varepsilon_i = 0, 1, -1 \text{ for } 1 \leqq i < j \right\}.$$

It is clear that if k_j grow fast enough

$$\frac{\partial (f * A)}{\partial \theta_{s'}} \approx 2\pi i \sum_{j=1}^{J} k_j \sigma_j^{(s')} \left\{ D_{j,+}(\bar{\theta}) \exp \left[2\pi i k_j \left(\sum_{s=1}^{2^J} \sigma_j^{(s)} \theta_s \right) \right] - D_{j,-}(\bar{\theta}) \exp \left[-2 i \pi k_j \left(\sum_{s=1}^{2^J} \sigma_j^{(s)} \theta_s \right) \right] \right\}.$$

Therefore, for fixed $\bar{\theta}$, by construction

$$\max_{s'} | \partial_{s'} (f * A) | \sim \sum_{j=1}^{J} k_j \, | D_{j,+}(\bar{\theta}) \exp[\cdots] - D_{j,-}(\bar{\theta}) \exp[-\cdots] |$$

and from (1)

$$\| f \|_W \geqq c \sum_{j=1}^{J} k_j \left| \int D_{j,+}(\bar{\theta}) d\bar{\theta} \right| = c \sum_{j=1}^{J} k_j \, | \widehat{f * A} (k_j \sigma_j^{(1)}, \ldots ; k_j \sigma_j^{(2^J)}) |. \tag{2}$$

Notice that from the definition of A

$$\widehat{f * A} (k_j \sigma_j^{(1)}, \ldots, k_j \sigma_j^{(2^J)}) = \tfrac{1}{2} \hat{f} (k_j \sigma_j^{(1)}, \ldots, k_j \sigma_j^{(2^J)}).$$

By (2), we proved that the map

$$W \to l_J^1 \colon f \to \{k_j \hat{f}(k_j\sigma_j^{(1)}, \ldots, k_j\sigma_j^{(2^J)})\}_{1 \leq j \leq J}$$

admits an absolute bound. Obviously $(1/k_j)\exp[2\pi i k_j(\sigma_j^{(1)}\theta_1 + \cdots + \sigma_j^{(2^J)}\theta_{2^J})]$ has norm 2π in W and is mapped on the jth unit-vector.

This completes the proof.

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